

Robust Regression and Lasso

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Abstract

Lasso, or ℓ^1 regularized least squares, has been explored extensively for its remarkable sparsity properties. It is shown in this paper that the solution to Lasso, in addition to its sparsity, has robustness properties: it is the solution to a robust optimization problem. This has two important consequences. First, robustness provides a connection of the regularizer to a physical property, namely, protection from noise. This allows a principled selection of the regularizer, and in particular, generalizations of Lasso that also yield convex optimization problems are obtained by considering different uncertainty sets.

Secondly, robustness can itself be used as an avenue to exploring different properties of the solution. In particular, it is shown that robustness of the solution explains why the solution is sparse. The analysis as well as the specific results obtained differ from standard sparsity results, providing different geometric intuition. Furthermore, it is shown that the robust optimization formulation is related to kernel density estimation, and based on this approach, a proof that Lasso is consistent is given using robustness directly. Finally, a theorem saying that sparsity and algorithmic stability contradict each other, and hence Lasso is not stable, is presented.

Index Terms

Statistical Learning, Regression, Regularization, Kernel density estimator, Lasso, Robustness, Sparsity, Stability.

I. INTRODUCTION

In this paper we consider linear regression problems with least-square error. The problem is to find a vector \mathbf{x} so that the ℓ_2 norm of the residual $\mathbf{b} - A\mathbf{x}$ is minimized, for a given matrix $A \in \mathbb{R}^{n \times m}$ and vector $\mathbf{b} \in \mathbb{R}^n$. From a learning/regression perspective, each row of A can be regarded as a training sample, and the corresponding element of \mathbf{b} as the target value of this observed sample. Each column of A corresponds to a feature, and the objective is to find a set of weights so that the weighted sum of the feature values approximates the target value.

It is well known that minimizing the least squared error can lead to sensitive solutions [1]–[4]. Many regularization methods have been proposed to decrease this sensitivity. Among them, Tikhonov regularization [5] and Lasso [6], [7] are two widely known and cited algorithms. These methods minimize a weighted sum of the residual norm and a certain regularization term, $\|\mathbf{x}\|_2$ for Tikhonov regularization and $\|\mathbf{x}\|_1$ for Lasso. In addition to providing regularity, Lasso is also known for the tendency to select sparse solutions. Recently this has attracted much attention for its ability to reconstruct sparse solutions when sampling occurs far below the Nyquist rate, and

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also for its ability to recover the sparsity pattern exactly with probability one, asymptotically as the number of observations increases (there is an extensive literature on this subject, and we refer the reader to [8]–[12] and references therein).

The first result of this paper is that the solution to Lasso has robustness properties: it is the solution to a robust optimization problem. In itself, this interpretation of Lasso as the solution to a robust least squares problem is a development in line with the results of [13]. There, the authors propose an alternative approach of reducing sensitivity of linear regression by considering a robust version of the regression problem, i.e., minimizing the worst-case residual for the observations under some unknown but bounded disturbance. Most of the research in this area considers either the case where the disturbance is row-wise uncoupled [14], or the case where the Frobenius norm of the disturbance matrix is bounded [13].

None of these robust optimization approaches produces a solution that has sparsity properties (in particular, the solution to Lasso does not solve any of these previously formulated robust optimization problems). In contrast, we investigate the robust regression problem where the uncertainty set is defined by feature-wise constraints. Such a noise model is of interest when values of features are obtained with some noisy pre-processing steps, and the magnitudes of such noises are known or bounded. Another situation of interest is where features are meaningfully coupled. We define *coupled* and *uncoupled* disturbances and uncertainty sets precisely in Section II-A below. Intuitively, a disturbance is feature-wise coupled if the variation or disturbance across features satisfy joint constraints, and uncoupled otherwise.

Considering the solution to Lasso as the solution of a robust least squares problem has two important consequences. First, robustness provides a connection of the regularizer to a physical property, namely, protection from noise. This allows more principled selection of the regularizer, and in particular, considering different uncertainty sets, we construct generalizations of Lasso that also yield convex optimization problems.

Secondly, and perhaps most significantly, robustness is a strong property that can itself be used as an avenue to investigating different properties of the solution. We show that robustness of the solution can explain why the solution is sparse. The analysis as well as the specific results we obtain differ from standard sparsity results, providing different geometric intuition, and extending beyond the least-squares setting. Sparsity results obtained for Lasso ultimately depend on the fact that introducing additional features incurs larger ℓ^1 -penalty than the least squares error reduction. In contrast, we exploit the fact that a robust solution is, by definition, the optimal solution under a worst-case perturbation. Our results show that, essentially, a coefficient of the solution is nonzero if the corresponding feature is relevant under all allowable perturbations. In addition to sparsity, we also use robustness directly to prove consistency of Lasso.

We briefly list the main contributions as well as the organization of this paper.

- In Section II, we formulate the robust regression problem with feature-wise independent disturbances, and show that this formulation is equivalent to a least-square problem with a weighted ℓ_1 norm regularization term. Hence, we provide an interpretation of Lasso from a robustness perspective.
- We generalize the robust regression formulation to loss functions of arbitrary norm in Section III. We also consider uncertainty sets that require disturbances of different features to satisfy joint conditions. This can be used to mitigate the conservativeness of the robust solution and to obtain solutions with additional properties.
- In Section IV, we present new sparsity results for the robust regression problem with feature-wise independent disturbances. This provides a new robustness-based explanation

to the sparsity of Lasso. Our approach gives new analysis and also geometric intuition, and furthermore allows one to obtain sparsity results for more general loss functions, beyond the squared loss.

- Next, we relate Lasso to kernel density estimation in Section V. This allows us to re-prove consistency in a statistical learning setup, using the new robustness tools and formulation we introduce. Along with our results on sparsity, this illustrates the power of robustness in explaining and also exploring different properties of the solution.
- Finally, we prove in Section VI a “no-free-lunch” theorem, stating that an algorithm that encourages sparsity cannot be stable.

Notation. We use capital letters to represent matrices, and boldface letters to represent column vectors. Row vectors are represented as the transpose of column vectors. For a vector \mathbf{z} , z_i denotes its i^{th} element. Throughout the paper, \mathbf{a}_i and \mathbf{r}_j^\top are used to denote the i^{th} column and the j^{th} row of the observation matrix A , respectively. We use a_{ij} to denote the ij element of A , hence it is the j^{th} element of \mathbf{r}_i , and i^{th} element of \mathbf{a}_j . For a convex function $f(\cdot)$, $\partial f(\mathbf{z})$ represents any of its sub-gradients evaluated at \mathbf{z} . A vector with length n and each element equals 1 is denoted as $\mathbf{1}_n$.

II. ROBUST REGRESSION WITH FEATURE-WISE DISTURBANCE

In this section, we show that our robust regression formulation recovers Lasso as a special case. We also derive probabilistic bounds that guide in the construction of the uncertainty set.

The regression formulation we consider differs from the standard Lasso formulation, as we minimize the norm of the error, rather than the squared norm. It is known that these two coincide up to a change of the regularization coefficient. Yet as we discuss above, our results lead to more flexible and potentially powerful robust formulations, and give new insight into known results.

A. Formulation

Robust linear regression considers the case where the observed matrix is corrupted by some potentially malicious disturbance. The objective is to find the optimal solution in the worst case sense. This is usually formulated as the following min-max problem,

Robust Linear Regression:

$$\min_{\mathbf{x} \in \mathbb{R}^m} \left\{ \max_{\Delta A \in \mathcal{U}} \|\mathbf{b} - (A + \Delta A)\mathbf{x}\|_2 \right\}, \quad (1)$$

where \mathcal{U} is called the *uncertainty set*, or the set of admissible disturbances of the matrix A . In this section, we consider the class of uncertainty sets that bound the norm of the disturbance to each feature, without placing any joint requirements across feature disturbances. That is, we consider the class of uncertainty sets:

$$\mathcal{U} \triangleq \left\{ (\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_m) \mid \|\boldsymbol{\delta}_i\|_2 \leq c_i, \ i = 1, \dots, m \right\}, \quad (2)$$

for given $c_i \geq 0$. We call these uncertainty sets *feature-wise uncoupled*, in contrast to *coupled* uncertainty sets that require disturbances of different features to satisfy some joint constraints (we discuss these extensively below, and their significance). While the inner maximization problem of (1) is nonconvex, we show in the next theorem that uncoupled norm-bounded uncertainty sets lead to an easily solvable optimization problem.

Theorem 1: The robust regression problem (1) with uncertainty set of the form (2) is equivalent to the following ℓ^1 regularized regression problem:

$$\min_{\mathbf{x} \in \mathbb{R}^m} \left\{ \|\mathbf{b} - A\mathbf{x}\|_2 + \sum_{i=1}^m c_i |x_i| \right\}. \quad (3)$$

Proof: Fix \mathbf{x}^* . We prove that $\max_{\Delta A \in \mathcal{U}} \|\mathbf{b} - (A + \Delta A)\mathbf{x}^*\|_2 = \|\mathbf{b} - A\mathbf{x}^*\|_2 + \sum_{i=1}^m c_i |x_i^*|$. The left hand side can be written as

$$\begin{aligned} & \max_{\Delta A \in \mathcal{U}} \|\mathbf{b} - (A + \Delta A)\mathbf{x}^*\|_2 \\ &= \max_{(\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_m) \|\boldsymbol{\delta}_i\|_2 \leq c_i} \left\| \mathbf{b} - (A + (\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_m))\mathbf{x}^* \right\|_2 \\ &= \max_{(\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_m) \|\boldsymbol{\delta}_i\|_2 \leq c_i} \left\| \mathbf{b} - A\mathbf{x}^* - \sum_{i=1}^m x_i^* \boldsymbol{\delta}_i \right\|_2 \\ &\leq \max_{(\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_m) \|\boldsymbol{\delta}_i\|_2 \leq c_i} \left\| \mathbf{b} - A\mathbf{x}^* \right\|_2 + \sum_{i=1}^m \|x_i^* \boldsymbol{\delta}_i\|_2 \\ &\leq \|\mathbf{b} - A\mathbf{x}^*\|_2 + \sum_{i=1}^m |x_i^*| c_i. \end{aligned} \quad (4)$$

Now, let

$$\mathbf{u} \triangleq \begin{cases} \frac{\mathbf{b} - A\mathbf{x}^*}{\|\mathbf{b} - A\mathbf{x}^*\|_2} & \text{if } A\mathbf{x}^* \neq \mathbf{b}, \\ \text{any vector with unit } \ell^2 \text{ norm} & \text{otherwise;} \end{cases}$$

and let

$$\boldsymbol{\delta}_i^* \triangleq -c_i \text{sgn}(x_i^*) \mathbf{u}.$$

Observe that $\|\boldsymbol{\delta}_i^*\|_2 \leq c_i$, hence $\Delta A^* \triangleq (\boldsymbol{\delta}_1^*, \dots, \boldsymbol{\delta}_m^*) \in \mathcal{U}$. Notice that

$$\begin{aligned} & \max_{\Delta A \in \mathcal{U}} \|\mathbf{b} - (A + \Delta A)\mathbf{x}^*\|_2 \\ &\geq \|\mathbf{b} - (A + \Delta A^*)\mathbf{x}^*\|_2 \\ &= \left\| \mathbf{b} - (A + (\boldsymbol{\delta}_1^*, \dots, \boldsymbol{\delta}_m^*))\mathbf{x}^* \right\|_2 \\ &= \left\| (\mathbf{b} - A\mathbf{x}^*) - \sum_{i=1}^m (-x_i^* c_i \text{sgn}(x_i^*) \mathbf{u}) \right\|_2 \\ &= \left\| (\mathbf{b} - A\mathbf{x}^*) + \left(\sum_{i=1}^m c_i |x_i^*| \right) \mathbf{u} \right\|_2 \\ &= \|\mathbf{b} - A\mathbf{x}^*\|_2 + \sum_{i=1}^m c_i |x_i^*|. \end{aligned} \quad (5)$$

The last equation holds from the definition of \mathbf{u} .

Combining Inequalities (4) and (5), establishes the equality $\max_{\Delta A \in \mathcal{U}} \|\mathbf{b} - (A + \Delta A)\mathbf{x}^*\|_2 = \|\mathbf{b} - A\mathbf{x}^*\|_2 + \sum_{i=1}^m c_i |x_i^*|$ for any \mathbf{x}^* . Minimizing over \mathbf{x} on both sides proves the theorem. ■
Taking $c_i = c$ and normalizing \mathbf{a}_i for all i , Problem (3) recovers the well-known Lasso [6], [7].

B. Uncertainty Set Construction

The selection of an uncertainty set \mathcal{U} in Robust Optimization is of fundamental importance. One way this can be done is as an approximation of so-called *chance constraints*, where a deterministic constraint is replaced by the requirement that a constraint is satisfied with at least some probability. These can be formulated when we know the distribution exactly, or when we have only partial information of the uncertainty, such as, e.g., first and second moments. This chance-constraint formulation is particularly important when the distribution has large support, rendering the naive robust optimization formulation overly pessimistic.

For confidence level η , the chance constraint formulation becomes:

$$\begin{aligned} & \text{minimize: } t \\ & \text{Subject to: } \Pr(\|\mathbf{b} - (A + \Delta A)\mathbf{x}\|_2 \leq t) \geq 1 - \eta. \end{aligned}$$

Here, \mathbf{x} and t are the decision variables.

Constructing the uncertainty set for feature i can be done quickly via line search and bisection, as long as we can evaluate $\Pr(\|\mathbf{a}_i\|_2 \geq c)$. If we know the distribution exactly (i.e., if we have complete probabilistic information), this can be quickly done via sampling. Another setting of interest is when we have access only to some moments of the distribution of the uncertainty, e.g., the mean and variance. In this setting, the uncertainty sets are constructed via a bisection procedure which evaluates the worst-case probability over all distributions with given mean and variance. We do this using a tight bound on the probability of an event, given the first two moments.

In the scalar case, the Markov Inequality provides such a bound. The next theorem is a generalization of the Markov inequality to \mathbb{R}^n , which bounds the probability where the disturbance on a given feature is more than c_i , if only the first and second moment of the random variable are known. We postpone the proof to the appendix, and refer the reader to [15] for similar results using semi-definite optimization.

Theorem 2: Consider a random vector $\mathbf{v} \in \mathbb{R}^n$, such that $\mathbb{E}(\mathbf{v}) = \mathbf{a}$, and $\mathbb{E}(\mathbf{v}\mathbf{v}^\top) = \Sigma$, $\Sigma \succeq 0$. Then we have

$$\Pr\{\|\mathbf{v}\|_2 \geq c_i\} \leq \begin{cases} \min_{P, \mathbf{q}, r, \lambda} & \text{Trace}(\Sigma P) + 2\mathbf{q}^\top \mathbf{a} + r \\ \text{subject to:} & \begin{pmatrix} P & \mathbf{q} \\ \mathbf{q}^\top & r \end{pmatrix} \succeq 0 \\ & \begin{pmatrix} I^{(m)} & \mathbf{0} \\ \mathbf{0}^\top & -c_i^2 \end{pmatrix} \preceq \lambda \begin{pmatrix} P & \mathbf{q} \\ \mathbf{q}^\top & r - 1 \end{pmatrix} \\ & \lambda \geq 0. \end{cases} \quad (6)$$

The optimization problem (6) is a semi-definite programming, which is known to be solved in polynomial time. Furthermore, if we replace $\mathbb{E}(\mathbf{v}\mathbf{v}^\top) = \Sigma$ by an inequality $\mathbb{E}(\mathbf{v}\mathbf{v}^\top) \leq \Sigma$, the uniform bound still holds. Thus, even if our estimation to the variance is not precise, we are still able to bound the probability of having “large” disturbance.

III. GENERAL UNCERTAINTY SETS

One reason the robust optimization formulation is powerful, is that having provided the connection to Lasso, it then allows the opportunity to generalize to efficient “Lasso-like” regularization algorithms.

In this section, we make several generalizations of the robust formulation (1) and derive counterparts of Theorem 1. We generalize the robust formulation in two ways: (a) to the case of arbitrary norm; and (b) to the case of coupled uncertainty sets.

We first consider the case of an arbitrary norm $\|\cdot\|_a$ of \mathbb{R}^n as a cost function rather than the squared loss. The proof of the next theorem is identical to that of Theorem 1, with only the ℓ^2 norm changed to $\|\cdot\|_a$.

Theorem 3: The robust regression problem

$$\min_{\mathbf{x} \in \mathbb{R}^m} \left\{ \max_{\Delta A \in \mathcal{U}_a} \|\mathbf{b} - (A + \Delta A)\mathbf{x}\|_a \right\}; \quad \mathcal{U}_a \triangleq \left\{ (\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_m) \mid \|\boldsymbol{\delta}_i\|_a \leq c_i, \quad i = 1, \dots, m \right\};$$

is equivalent to the following regularized regression problem

$$\min_{\mathbf{x} \in \mathbb{R}^m} \left\{ \|\mathbf{b} - A\mathbf{x}\|_a + \sum_{i=1}^m c_i |x_i| \right\}.$$

We next remove the assumption that the disturbances are feature-wise uncoupled. Allowing coupled uncertainty sets is useful when we have some additional information about potential noise in the problem, and we want to limit the conservativeness of the worst-case formulation. Consider the following uncertainty set:

$$\mathcal{U}' \triangleq \left\{ (\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_m) \mid f_j(\|\boldsymbol{\delta}_1\|_a, \dots, \|\boldsymbol{\delta}_m\|_a) \leq 0; \quad j = 1, \dots, k \right\},$$

where $f_j(\cdot)$ are convex functions. Notice that, both k and f_j can be arbitrary, hence this is a very general formulation, and provides us with significant flexibility in designing uncertainty sets and equivalently new regression algorithms (see for example Corollary 1 and 2). The following theorem converts this formulation to tractable optimization problems. The proof is postponed to the appendix.

Theorem 4: Assume that the set

$$\mathcal{Z} \triangleq \left\{ \mathbf{z} \in \mathbb{R}^m \mid f_j(\mathbf{z}) \leq 0, \quad j = 1, \dots, k; \quad \mathbf{z} \geq \mathbf{0} \right\}$$

has non-empty relative interior. Then the robust regression problem

$$\min_{\mathbf{x} \in \mathbb{R}^m} \left\{ \max_{\Delta A \in \mathcal{U}'} \|\mathbf{b} - (A + \Delta A)\mathbf{x}\|_a \right\}$$

is equivalent to the following regularized regression problem

$$\min_{\boldsymbol{\lambda} \in \mathbb{R}_+^k, \boldsymbol{\kappa} \in \mathbb{R}_+^m, \mathbf{x} \in \mathbb{R}^m} \left\{ \|\mathbf{b} - A\mathbf{x}\|_a + v(\boldsymbol{\lambda}, \boldsymbol{\kappa}, \mathbf{x}) \right\}; \quad (7)$$

$$\text{where: } v(\boldsymbol{\lambda}, \boldsymbol{\kappa}, \mathbf{x}) \triangleq \max_{\mathbf{c} \in \mathbb{R}^m} \left[(\boldsymbol{\kappa} + |\mathbf{x}|)^\top \mathbf{c} - \sum_{j=1}^k \lambda_j f_j(\mathbf{c}) \right]$$

Remark: Problem (7) is efficiently solvable. Denote $z^c(\boldsymbol{\lambda}, \boldsymbol{\kappa}, \mathbf{x}) \triangleq \left[(\boldsymbol{\kappa} + |\mathbf{x}|)^\top \mathbf{c} - \sum_{j=1}^k \lambda_j f_j(\mathbf{c}) \right]$. This is a convex function of $(\boldsymbol{\lambda}, \boldsymbol{\kappa}, \mathbf{x})$, and the sub-gradient of $z^c(\cdot)$ can be computed easily for any \mathbf{c} . The function $v(\boldsymbol{\lambda}, \boldsymbol{\kappa}, \mathbf{x})$ is the maximum of a set of convex functions, $z^c(\cdot)$, hence is convex, and satisfies

$$\partial v(\boldsymbol{\lambda}^*, \boldsymbol{\kappa}^*, \mathbf{x}^*) = \partial z^{c^0}(\boldsymbol{\lambda}^*, \boldsymbol{\kappa}^*, \mathbf{x}^*),$$

where \mathbf{c}_0 maximizes $\left[(\boldsymbol{\kappa}^* + |\mathbf{x}|^*)^\top \mathbf{c} - \sum_{j=1}^k \lambda_j^* f_j(\mathbf{c}) \right]$. We can efficiently evaluate \mathbf{c}_0 due to convexity of $f_j(\cdot)$, and hence we can efficiently evaluate the sub-gradient of $v(\cdot)$.

The next two corollaries are a direct application of Theorem 4.

Corollary 1: Suppose $\mathcal{U}' = \left\{ (\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_m) \mid \|\boldsymbol{\delta}_1\|_a, \dots, \|\boldsymbol{\delta}_m\|_a\|_s \leq l; \right\}$ for a symmetric norm $\|\cdot\|_s$, then the resulting regularized regression problem is

$$\min_{\mathbf{x} \in \mathbb{R}^m} \left\{ \|\mathbf{b} - A\mathbf{x}\|_a + l\|\mathbf{x}\|_s^* \right\}; \quad \text{where } \|\cdot\|_s^* \text{ is the dual norm of } \|\cdot\|_s.$$

This corollary interprets *arbitrary* norm-based regularizers from a robust regression perspective. For example, it is straightforward to show that if we take both $\|\cdot\|_a$ and $\|\cdot\|_s$ as the Euclidean norm, then \mathcal{U}' is the set of matrices with their Frobenious norms bounded, and Corollary 1 reduces to the robust formulation introduced by [13].

Corollary 2: Suppose $\mathcal{U}' = \left\{ (\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_m) \mid \exists \mathbf{c} \geq \mathbf{0} : T\mathbf{c} \leq \mathbf{s}; \|\boldsymbol{\delta}_j\|_a \leq c_j; \right\}$, then the resulting regularized regression problem is

$$\begin{aligned} \text{Minimize: } & \|\mathbf{b} - A\mathbf{x}\|_a + \mathbf{s}^\top \boldsymbol{\lambda} \\ \text{Subject to: } & \mathbf{x} \leq T^\top \boldsymbol{\lambda} \\ & -\mathbf{x} \leq T^\top \boldsymbol{\lambda} \\ & \boldsymbol{\lambda} \geq \mathbf{0}. \end{aligned}$$

Unlike previous results, this corollary considers general polytope uncertainty sets. Advantages of such sets include the linearity of the final formulation. Moreover, the modeling power is considerable, as many interesting disturbances can be modeled in this way.

We briefly mention some further examples meant to illustrate the power and flexibility of the robust formulation. We refer the interested reader to [16] for full details.

As the results above indicate, the robust formulation can model a broad class of uncertainties, and yield computationally tractable (i.e., convex) problems. In particular, one can use the polytope uncertainty discussed above, to show (see [16]) that by employing an uncertainty set first used in [17], we can model cardinality constrained noise, where some (unknown) subset of at most k features can be corrupted.

Another avenue one may take using robustness, and which is also possible to solve easily, is the case where the uncertainty set allows independent perturbation of the columns and the rows of the matrix A . The resulting formulation resembles the elastic-net formulation [18], where there is a combination of ℓ^2 and ℓ^1 regularization.

IV. SPARSITY

In this section, we investigate the sparsity properties of robust regression (1), and equivalently Lasso. Lasso's ability to recover sparse solutions has been extensively studied and discussed (cf [8]–[11]). There are generally two approaches. The first approach investigates the problem from a statistical perspective. That is, it assumes that the observations are generated by a (sparse) linear combination of the features, and investigates the asymptotic or probabilistic conditions required for Lasso to correctly recover the generative model. The second approach treats the problem from an optimization perspective, and studies under what conditions a pair (A, \mathbf{b}) defines a problem with sparse solutions (e.g., [19]).

We follow the second approach and do not assume a generative model. Instead, we consider the conditions that lead to a feature receiving zero weight. Our first result paves the way for the remainder of this section. We show in Theorem 5 that, essentially, a feature receives no weight (namely, $x_i^* = 0$) if there exists an allowable perturbation of that feature which makes it irrelevant. This result holds for general norm loss functions, but in the ℓ^2 case, we obtain further geometric results. For instance, using Theorem 5, we show, among other results, that “nearly” orthogonal features get zero weight (Theorem 6). Using similar tools, we provide additional results in [16]. There, we show, among other results, that the sparsity pattern of any optimal solution must satisfy certain angular separation conditions between the residual and the relevant features, and that “nearly” linearly dependent features get zero weight.

Substantial research regarding sparsity properties of Lasso can be found in the literature (cf [8]–[11], [20]–[23] and many others). In particular, similar results as in point (a), that rely on an *incoherence* property, have been established in, e.g., [19], and are used as standard tools in investigating sparsity of Lasso from the statistical perspective. However, a proof exploiting robustness and properties of the uncertainty is novel. Indeed, such a proof shows a fundamental connection between robustness and sparsity, and implies that robustifying w.r.t. a feature-wise independent uncertainty set might be a plausible way to achieve sparsity for other problems.

To state the main theorem of this section, from which the other results derive, we introduce some notation to facilitate the discussion. Given a feature-wise uncoupled uncertainty set, \mathcal{U} , an index subset $I \subseteq \{1, \dots, n\}$, and any $\Delta A \in \mathcal{U}$, let ΔA^I denote the element of \mathcal{U} that equals ΔA on each feature indexed by $i \in I$, and is zero elsewhere. Then, we can write any element $\Delta A \in \mathcal{U}$ as $\Delta A^I + \Delta A^{I^c}$ (where $I^c = \{1, \dots, n\} \setminus I$). Then we have the following theorem. We note that the result holds for any norm loss function, but we state and prove it for the ℓ^2 norm, since the proof for other norms is identical.

Theorem 5: The robust regression problem

$$\min_{\mathbf{x} \in \mathbb{R}^m} \left\{ \max_{\Delta A \in \mathcal{U}} \|\mathbf{b} - (A + \Delta A)\mathbf{x}\|_2 \right\},$$

has a solution supported on an index set I if there exists some perturbation $\tilde{\Delta A}^{I^c} \in \mathcal{U}$ of the features in I^c , such that the robust regression problem

$$\min_{\mathbf{x} \in \mathbb{R}^m} \left\{ \max_{\tilde{\Delta A}^I \in \mathcal{U}^I} \|\mathbf{b} - (A + \tilde{\Delta A}^{I^c} + \tilde{\Delta A}^I)\mathbf{x}\|_2 \right\},$$

has a solution supported on the set I .

Thus, a robust regression has an optimal solution supported on a set I , if *any* perturbation of the features corresponding to the complement of I makes them irrelevant. Theorem 5 is a special case of the following theorem with $c_j = 0$ for all $j \notin I$:

Theorem 5’. Let \mathbf{x}^* be an optimal solution of the robust regression problem:

$$\min_{\mathbf{x} \in \mathbb{R}^m} \left\{ \max_{\Delta A \in \mathcal{U}} \|\mathbf{b} - (A + \Delta A)\mathbf{x}\|_2 \right\},$$

and let $I \subseteq \{1, \dots, m\}$ be such that $x_j^* = 0 \forall j \notin I$. Let

$$\tilde{\mathcal{U}} \triangleq \left\{ (\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_m) \mid \|\boldsymbol{\delta}_i\|_2 \leq c_i, \ i \in I; \ \|\boldsymbol{\delta}_j\|_2 \leq c_j + l_j, \ j \notin I \right\}.$$

Then, \mathbf{x}^* is an optimal solution of

$$\min_{\mathbf{x} \in \mathbb{R}^m} \left\{ \max_{\Delta A \in \tilde{\mathcal{U}}} \|\mathbf{b} - (\tilde{A} + \Delta A)\mathbf{x}\|_2 \right\},$$

for any \tilde{A} that satisfies $\|\tilde{\mathbf{a}}_j - \mathbf{a}_j\| \leq l_j$ for $j \notin I$, and $\tilde{\mathbf{a}}_i = \mathbf{a}_i$ for $i \in I$.

Proof: Notice that

$$\begin{aligned} & \max_{\Delta A \in \tilde{\mathcal{U}}} \left\| \mathbf{b} - (A + \Delta A)\mathbf{x}^* \right\|_2 \\ &= \max_{\Delta A \in \mathcal{U}} \left\| \mathbf{b} - (A + \Delta A)\mathbf{x}^* \right\|_2 \\ &= \max_{\Delta A \in \mathcal{U}} \left\| \mathbf{b} - (\tilde{A} + \Delta A)\mathbf{x}^* \right\|_2. \end{aligned}$$

These equalities hold because for $j \notin I$, $x_j^* = 0$, hence the j^{th} column of both \tilde{A} and ΔA has no effect on the residual.

For an arbitrary \mathbf{x}' , we have

$$\begin{aligned} & \max_{\Delta A \in \tilde{\mathcal{U}}} \left\| \mathbf{b} - (A + \Delta A)\mathbf{x}' \right\|_2 \\ & \geq \max_{\Delta A \in \mathcal{U}} \left\| \mathbf{b} - (\tilde{A} + \Delta A)\mathbf{x}' \right\|_2. \end{aligned}$$

This is because, $\|\mathbf{a}_j - \tilde{\mathbf{a}}_j\| \leq l_j$ for $j \notin I$, and $\mathbf{a}_i = \tilde{\mathbf{a}}_i$ for $i \in I$. Hence, we have

$$\{A + \Delta A \mid \Delta A \in \mathcal{U}\} \subseteq \{\tilde{A} + \Delta A \mid \Delta A \in \tilde{\mathcal{U}}\}.$$

Finally, notice that

$$\max_{\Delta A \in \mathcal{U}} \left\| \mathbf{b} - (A + \Delta A)\mathbf{x}^* \right\|_2 \leq \max_{\Delta A \in \mathcal{U}} \left\| \mathbf{b} - (A + \Delta A)\mathbf{x}' \right\|_2.$$

Therefore we have

$$\max_{\Delta A \in \tilde{\mathcal{U}}} \left\| \mathbf{b} - (\tilde{A} + \Delta A)\mathbf{x}^* \right\|_2 \leq \max_{\Delta A \in \tilde{\mathcal{U}}} \left\| \mathbf{b} - (\tilde{A} + \Delta A)\mathbf{x}' \right\|_2.$$

Since this holds for arbitrary \mathbf{x}' , we establish the theorem. ■

We can interpret the result of this theorem by considering a generative model¹ $b = \sum_{i \in I} w_i a_i + \tilde{\xi}$ where $I \subseteq \{1, \dots, m\}$ and $\tilde{\xi}$ is a random variable, i.e., b is generated by features belonging to I . In this case, for a feature $j \notin I$, Lasso would assign zero weight as long as there exists a perturbed value of this feature, such that the optimal regression assigned it zero weight.

When we consider ℓ^2 loss, we can translate the condition of a feature being “irrelevant” into a geometric condition, namely, orthogonality. We now use the result of Theorem 5 to show that robust regression has a sparse solution as long as an incoherence-type property is satisfied. This result is more in line with the traditional sparsity results, but we note that the geometric reasoning is different, and ours is based on robustness. Indeed, we show that a feature receives zero weight, if it is “nearly” (i.e., within an allowable perturbation) orthogonal to the signal, and

¹While we are not assuming generative models to establish the results, it is still interesting to see how these results can help in a generative model setup.

all relevant features.

Theorem 6: Let $c_i = c$ for all i and consider ℓ^2 loss. If there exists $I \subset \{1, \dots, m\}$ such that for all $\mathbf{v} \in \text{span}(\{\mathbf{a}_i, i \in I\} \cup \{\mathbf{b}\})$, $\|\mathbf{v}\| = 1$, we have $\mathbf{v}^\top \mathbf{a}_j \leq c$, $\forall j \notin I$, then any optimal solution \mathbf{x}^* satisfies $x_j^* = 0$, $\forall j \notin I$.

Proof: For $j \notin I$, let \mathbf{a}_j^- denote the projection of \mathbf{a}_j onto the span of $\{\mathbf{a}_i, i \in I\} \cup \{\mathbf{b}\}$, and let $\mathbf{a}_j^+ \triangleq \mathbf{a}_j - \mathbf{a}_j^-$. Thus, we have $\|\mathbf{a}_j^-\| \leq c$. Let \hat{A} be such that

$$\hat{\mathbf{a}}_i = \begin{cases} \mathbf{a}_i & i \in I; \\ \mathbf{a}_i^+ & i \notin I. \end{cases}$$

Now let

$$\hat{\mathcal{U}} \triangleq \{(\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_m) \mid \|\boldsymbol{\delta}_i\|_2 \leq c, i \in I; \|\boldsymbol{\delta}_j\|_2 = 0, j \notin I\}.$$

Consider the robust regression problem $\min_{\hat{\mathbf{x}}} \left\{ \max_{\Delta A \in \hat{\mathcal{U}}} \|\mathbf{b} - (\hat{A} + \Delta A)\hat{\mathbf{x}}\|_2 \right\}$, which is equivalent to $\min_{\hat{\mathbf{x}}} \left\{ \|\mathbf{b} - \hat{A}\hat{\mathbf{x}}\|_2 + \sum_{i \in I} c|\hat{x}_i| \right\}$. Note that the $\hat{\mathbf{a}}_j$ are orthogonal to the span of $\{\hat{\mathbf{a}}_i, i \in I\} \cup \{\mathbf{b}\}$. Hence for any given $\hat{\mathbf{x}}$, by changing \hat{x}_j to zero for all $j \notin I$, the minimizing objective does not increase.

Since $\|\hat{\mathbf{a}} - \hat{\mathbf{a}}_j\| = \|\mathbf{a}_j^-\| \leq c \forall j \notin I$, (and recall that $\mathcal{U} = \{(\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_m) \mid \|\boldsymbol{\delta}_i\|_2 \leq c, \forall i\}$) applying Theorem 5 concludes the proof. \blacksquare

V. DENSITY ESTIMATION AND CONSISTENCY

In this section, we investigate the robust linear regression formulation from a statistical perspective and rederive *using only robustness properties* that Lasso is asymptotically consistent. The basic idea of the consistency proof is as follows. We show that the robust optimization formulation can be seen to be the maximum error w.r.t. a class of probability measures. This class includes a kernel density estimator, and using this, we show that Lasso is consistent.

A. Robust Optimization, Worst-case Expected Utility and Kernel Density Estimator

In this subsection, we present some notions and intermediate results. In particular, we link a robust optimization formulation with a worst expected utility (w.r.t. a class of probability measures); we then briefly recall the definition of a kernel density estimator. Such results will be used in establishing the consistency of Lasso, as well as providing some additional insights on robust optimization. Proofs are postponed to the appendix.

We first establish a general result on the equivalence between a robust optimization formulation and a worst-case expected utility:

Proposition 1: Given a function $g : \mathbb{R}^{m+1} \rightarrow \mathbb{R}$ and Borel sets $\mathcal{Z}_1, \dots, \mathcal{Z}_n \subseteq \mathbb{R}^{m+1}$, let

$$\mathcal{P}_n \triangleq \{\mu \in \mathcal{P} \mid \forall S \subseteq \{1, \dots, n\} : \mu(\bigcup_{i \in S} \mathcal{Z}_i) \geq |S|/n\}.$$

The following holds

$$\frac{1}{n} \sum_{i=1}^n \sup_{(\mathbf{r}_i, b_i) \in \mathcal{Z}_i} h(\mathbf{r}_i, b_i) = \sup_{\mu \in \mathcal{P}_n} \int_{\mathbb{R}^{m+1}} h(\mathbf{r}, b) d\mu(\mathbf{r}, b).$$

This leads to the following corollary for Lasso, which states that for a given \mathbf{x} , the robust regression loss over the training data is equal to the worst-case expected *generalization error*.

Corollary 3: Given $\mathbf{b} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times m}$, the following equation holds for any $\mathbf{x} \in \mathbb{R}^m$,

$$\|\mathbf{b} - A\mathbf{x}\|_2 + \sqrt{n}c_n\|\mathbf{x}\|_1 + \sqrt{n}c_n = \sup_{\mu \in \hat{\mathcal{P}}(n)} \sqrt{n \int_{\mathbb{R}^{m+1}} (b' - \mathbf{r}'^\top \mathbf{x})^2 d\mu(\mathbf{r}', b')}. \quad (8)$$

Here²,

$$\begin{aligned} \hat{\mathcal{P}}(n) &\triangleq \bigcup_{\|\boldsymbol{\sigma}\|_2 \leq \sqrt{n}c_n; \forall i: \|\boldsymbol{\delta}_i\|_2 \leq \sqrt{n}c_n} \mathcal{P}_n(A, \Delta, \mathbf{b}, \boldsymbol{\sigma}); \\ \mathcal{P}_n(A, \Delta, \mathbf{b}, \boldsymbol{\sigma}) &\triangleq \{\mu \in \mathcal{P} | \mathcal{Z}_i = [b_i - \sigma_i, b_i + \sigma_i] \times \prod_{j=1}^m [a_{ij} - \delta_{ij}, a_{ij} + \delta_{ij}]; \\ &\quad \forall S \subseteq \{1, \dots, n\} : \mu(\bigcup_{i \in S} \mathcal{Z}_i) \geq |S|/n\}. \end{aligned}$$

Remark 1: We briefly explain Corollary 3 to avoid possible confusions. Equation (8) is a non-probabilistic equality. That is, it holds without any assumption (e.g., i.i.d. or generated by certain distributions) on \mathbf{b} and A . And it does not involve any probabilistic operation such as taking expectation on the left-hand-side, instead, it is an equivalence relationship which hold for an arbitrary set of samples. Notice that, the right-hand-side also depends on the samples since $\hat{\mathcal{P}}(n)$ is defined through A and \mathbf{b} . Indeed, $\hat{\mathcal{P}}(n)$ represents the union of classes of distributions $\mathcal{P}_n(A, \Delta, \mathbf{b}, \boldsymbol{\sigma})$ such that the norm of each column of Δ is bounded, where $\mathcal{P}_n(A, \Delta, \mathbf{b}, \boldsymbol{\sigma})$ is the set of distributions corresponds to (see Proposition 1) disturbance in hyper-rectangle Borel sets $\mathcal{Z}_1, \dots, \mathcal{Z}_n$ centered at (b_i, \mathbf{r}_i^\top) with lengths $(2\sigma_i, 2\delta_{i1}, \dots, 2\delta_{im})$.

We will later show that $\hat{\mathcal{P}}_n$ consists a kernel density estimator. Hence we recall here its definition. The *kernel density estimator* for a density \hat{h} in \mathbb{R}^d , originally proposed in [24], [25], is defined by

$$h_n(\mathbf{x}) = (nc_n^d)^{-1} \sum_{i=1}^n K\left(\frac{\mathbf{x} - \hat{\mathbf{x}}_i}{c_n}\right),$$

where $\{c_n\}$ is a sequence of positive numbers, $\hat{\mathbf{x}}_i$ are i.i.d. samples generated according to \hat{f} , and K is a Borel measurable function (kernel) satisfying $K \geq 0$, $\int K = 1$. See [26], [27] and the reference therein for detailed discussions. Figure 1 illustrates a kernel density estimator using Gaussian kernel for a randomly generated sample-set. A celebrated property of a kernel density estimator is that it converges in \mathcal{L}^1 to \hat{h} when $c_n \downarrow 0$ and $nc_n^d \uparrow \infty$ [26].

B. Consistency of Lasso

We restrict our discussion to the case where the magnitude of the allowable uncertainty for all features equals c , (i.e., the standard Lasso) and establish the statistical consistency of Lasso from a distributional robustness argument. Generalization to the non-uniform case is straightforward. Throughout, we use c_n to represent c where there are n samples (we take c_n to zero).

Recall the standard generative model in statistical learning: let \mathbb{P} be a probability measure with bounded support that generates i.i.d samples (b_i, \mathbf{r}_i) , and has a density $f^*(\cdot)$. Denote the

²Recall that a_{ij} is the j^{th} element of \mathbf{r}_i

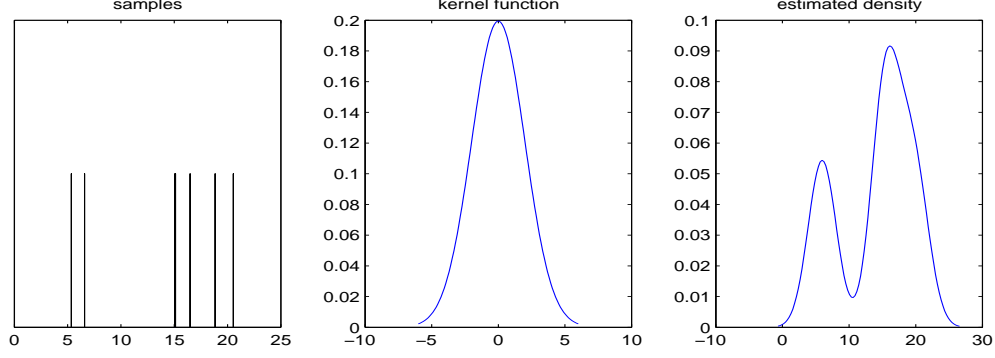


Fig. 1. Illustration of Kernel Density Estimation.

set of the first n samples by \mathcal{S}_n . Define

$$\begin{aligned} \mathbf{x}(c_n, \mathcal{S}_n) &\triangleq \arg \min_{\mathbf{x}} \left\{ \sqrt{\frac{1}{n} \sum_{i=1}^n (b_i - \mathbf{r}_i^\top \mathbf{x})^2 + c_n \|\mathbf{x}\|_1} \right\} \\ &= \arg \min_{\mathbf{x}} \left\{ \frac{\sqrt{n}}{n} \sqrt{\sum_{i=1}^n (b_i - \mathbf{r}_i^\top \mathbf{x})^2 + c_n \|\mathbf{x}\|_1} \right\}; \\ \mathbf{x}(\mathbb{P}) &\triangleq \arg \min_{\mathbf{x}} \left\{ \sqrt{\int_{b, \mathbf{r}} (b - \mathbf{r}^\top \mathbf{x})^2 d\mathbb{P}(b, \mathbf{r})} \right\}. \end{aligned}$$

In words, $\mathbf{x}(c_n, \mathcal{S}_n)$ is the solution to Lasso with the tradeoff parameter set to $c_n \sqrt{n}$, and $\mathbf{x}(\mathbb{P})$ is the “true” optimal solution. We have the following consistency result. The theorem itself is a well-known result. However, the proof technique is novel. This technique is of interest because the standard techniques to establish consistency in statistical learning including Vapnik-Chervonenkis (VC) dimension (e.g., [28]) and algorithmic stability (e.g., [29]) often work for a limited range of algorithms, e.g., the k -Nearest Neighbor is known to have infinite VC dimension, and we show in Section VI that *Lasso is not stable*. In contrast, a much wider range of algorithms have robustness interpretations, allowing a unified approach to prove their consistency.

Theorem 7: Let $\{c_n\}$ be such that $c_n \downarrow 0$ and $\lim_{n \rightarrow \infty} n(c_n)^{m+1} = \infty$. Suppose there exists a constant H such that $\|\mathbf{x}(c_n, \mathcal{S}_n)\|_2 \leq H$. Then,

$$\lim_{n \rightarrow \infty} \sqrt{\int_{b, \mathbf{r}} (b - \mathbf{r}^\top \mathbf{x}(c_n, \mathcal{S}_n))^2 d\mathbb{P}(b, \mathbf{r})} = \sqrt{\int_{b, \mathbf{r}} (b - \mathbf{r}^\top \mathbf{x}(\mathbb{P}))^2 d\mathbb{P}(b, \mathbf{r})},$$

almost surely.

Proof: Step 1: We show that the right hand side of Equation (8) includes a kernel density estimator for the true (unknown) distribution. Consider the following kernel estimator given

samples $\mathcal{S}_n = (b_i, \mathbf{r}_i)_{i=1}^n$ and tradeoff parameter c_n ,

$$f_n(b, \mathbf{r}) \triangleq (nc_n^{m+1})^{-1} \sum_{i=1}^n K\left(\frac{b - b_i, \mathbf{r} - \mathbf{r}_i}{c_n}\right), \quad (9)$$

where: $K(\mathbf{x}) \triangleq I_{[-1, +1]^{m+1}}(\mathbf{x})/2^{m+1}$.

Let $\hat{\mu}_n$ denote the distribution given by the density function $f_n(b, \mathbf{r})$. Easy to check that $\hat{\mu}_n$ belongs to $\mathcal{P}_n(A, (c_n \mathbf{1}_n, \dots, c_n \mathbf{1}_n), \mathbf{b}, c_n \mathbf{1}_n)$ and hence belongs to $\hat{\mathcal{P}}(n)$ by definition.

Step 2: Using the \mathcal{L}^1 convergence property of the kernel density estimator, we prove the consistency of robust regression and equivalently Lasso.

First notice that, $\|\mathbf{x}(c_n, \mathcal{S}_n)\|_2 \leq H$ and \mathbb{P} has a bounded support implies that there exists a universal constant C such that

$$\max_{b, \mathbf{r}} (b - \mathbf{r}^\top \mathbf{w}(c_n, \mathcal{S}_n))^2 \leq C.$$

By Corollary 3 and $\hat{\mu}_n \in \hat{\mathcal{P}}(n)$ we have

$$\begin{aligned} & \sqrt{\int_{b, \mathbf{r}} (b - \mathbf{r}^\top \mathbf{x}(c_n, \mathcal{S}_n))^2 d\hat{\mu}_n(b, \mathbf{r})} \\ & \leq \sup_{\mu \in \hat{\mathcal{P}}(n)} \sqrt{\int_{b, \mathbf{r}} (b - \mathbf{r}^\top \mathbf{x}(c_n, \mathcal{S}_n))^2 d\mu(b, \mathbf{r})} \\ & = \frac{\sqrt{n}}{n} \sqrt{\sum_{i=1}^n (b_i - \mathbf{r}_i^\top \mathbf{x}(c_n, \mathcal{S}_n))^2 + c_n \|\mathbf{x}(c_n, \mathcal{S}_n)\|_1 + c_n} \\ & \leq \frac{\sqrt{n}}{n} \sqrt{\sum_{i=1}^n (b_i - \mathbf{r}_i^\top \mathbf{x}(\mathbb{P}))^2 + c_n \|\mathbf{x}(\mathbb{P})\|_1 + c_n}, \end{aligned}$$

the last inequality holds by definition of $\mathbf{x}(c_n, \mathcal{S}_n)$.

Taking the square of both sides, we have

$$\begin{aligned} & \int_{b, \mathbf{r}} (b - \mathbf{r}^\top \mathbf{x}(c_n, \mathcal{S}_n))^2 d\hat{\mu}_n(b, \mathbf{r}) \\ & \leq \frac{1}{n} \sum_{i=1}^n (b_i - \mathbf{r}_i^\top \mathbf{x}(\mathbb{P}))^2 + c_n^2 (1 + \|\mathbf{x}(\mathbb{P})\|_1)^2 \\ & \quad + 2c_n (1 + \|\mathbf{x}(\mathbb{P})\|_1) \sqrt{\frac{1}{n} \sum_{i=1}^n (b_i - \mathbf{r}_i^\top \mathbf{x}(\mathbb{P}))^2}. \end{aligned}$$

Notice that, the right-hand side converges to $\int_{b, \mathbf{r}} (b - \mathbf{r}^\top \mathbf{x}(\mathbb{P}))^2 d\mathbb{P}(b, \mathbf{r})$ as $n \uparrow \infty$ and $c_n \downarrow 0$

almost surely. Furthermore, we have

$$\begin{aligned}
& \int_{b, \mathbf{r}} (b - \mathbf{r}^\top \mathbf{x}(c_n, \mathcal{S}_n))^2 d\mathbb{P}(b, \mathbf{r}) \\
& \leq \int_{b, \mathbf{r}} (b - \mathbf{r}^\top \mathbf{x}(c_n, \mathcal{S}_n))^2 d\hat{\mu}_n(b, \mathbf{r}) \\
& \quad + \left[\max_{b, \mathbf{r}} (b - \mathbf{r}^\top \mathbf{x}(c_n, \mathcal{S}_n))^2 \right] \int_{b, \mathbf{r}} |f_n(b, \mathbf{r}) - f^*(b, \mathbf{r})| d(b, \mathbf{r}) \\
& \leq \int_{b, \mathbf{r}} (b - \mathbf{r}^\top \mathbf{x}(c_n, \mathcal{S}_n))^2 d\hat{\mu}_n(b, \mathbf{r}) + C \int_{b, \mathbf{r}} |f_n(b, \mathbf{r}) - f^*(b, \mathbf{r})| d(b, \mathbf{r}),
\end{aligned}$$

where the last inequality follows from the definition of C . Notice that $\int_{b, \mathbf{r}} |f_n(b, \mathbf{r}) - f^*(b, \mathbf{r})| d(b, \mathbf{r})$ goes to zero almost surely when $c_n \downarrow 0$ and $nc_n^{m+1} \uparrow \infty$ since $f_n(\cdot)$ is a kernel density estimation of $f^*(\cdot)$ (see e.g. Theorem 3.1 of [26]). Hence the theorem follows. \blacksquare

We can remove the assumption that $\|\mathbf{x}(c_n, \mathcal{S}_n)\|_2 \leq H$, and as in Theorem 7, the proof technique rather than the result itself is of interest.

Theorem 8: Let $\{c_n\}$ converge to zero sufficiently slowly. Then

$$\lim_{n \rightarrow \infty} \sqrt{\int_{b, \mathbf{r}} (b - \mathbf{r}^\top \mathbf{x}(c_n, \mathcal{S}_n))^2 d\mathbb{P}(b, \mathbf{r})} = \sqrt{\int_{b, \mathbf{r}} (b - \mathbf{r}^\top \mathbf{x}(\mathbb{P}))^2 d\mathbb{P}(b, \mathbf{r})},$$

almost surely.

Proof: To prove the theorem, we need to consider a set of distributions belonging to $\hat{\mathcal{P}}(n)$. Hence we establish the following lemma first.

Lemma 1: Partition the support of \mathbb{P} as V_1, \dots, V_T such the ℓ^∞ radius of each set is less than c_n . If a distribution μ satisfies

$$\mu(V_t) = \left| \{i | (b_i, \mathbf{r}_i) \in V_t\} \right| / n; \quad t = 1, \dots, T, \quad (10)$$

then $\mu \in \hat{\mathcal{P}}(n)$.

Proof: Let $\mathcal{Z}_i = [b_i - c_n, b_i + c_n] \times \prod_{j=1}^m [a_{ij} - c_n, a_{ij} + c_n]$; recall that a_{ij} the j^{th} element of \mathbf{r}_i . Notice V_t has ℓ^∞ norm less than c_n we have

$$(b_i, \mathbf{r}_i \in V_t) \Rightarrow V_t \subseteq \mathcal{Z}_i.$$

Therefore, for any $S \subseteq \{1, \dots, n\}$, the following holds

$$\begin{aligned}
\mu\left(\bigcup_{i \in S} \mathcal{Z}_i\right) & \geq \mu\left(\bigcup_{i \in S} V_t | \exists i \in S : b_i, \mathbf{r}_i \in V_t\right) \\
& = \sum_{t | \exists i \in S : b_i, \mathbf{r}_i \in V_t} \mu(V_t) = \sum_{t | \exists i \in S : b_i, \mathbf{r}_i \in V_t} \#((b_i, \mathbf{r}_i) \in V_t) / n \geq |S| / n.
\end{aligned}$$

Hence $\mu \in \mathcal{P}_n(A, \Delta, b, c_n)$ where each element of Δ is c_n , which leads to $\mu \in \hat{\mathcal{P}}(n)$. \blacksquare

Now we proceed to prove the theorem. Partition the support of \mathbb{P} into T subsets such that ℓ^∞ radius of each one is smaller than c_n . Denote $\tilde{\mathcal{P}}(n)$ as the set of probability measures satisfying Equation (10). Hence $\tilde{\mathcal{P}}(n) \subseteq \hat{\mathcal{P}}(n)$ by Lemma 1. Further notice that there exists a universal constant K such that $\|\mathbf{x}(c_n, \mathcal{S}_n)\|_2 \leq K/c_n$ due to the fact that the square loss of the solution

$\mathbf{x} = \mathbf{0}$ is bounded by a constant only depends on the support of \mathbb{P} . Thus, there exists a constant C such that $\max_{b, \mathbf{r}} (b - \mathbf{r}^\top \mathbf{x}(c_n, \mathcal{S}_n))^2 \leq C/c_n^2$.

Follow a similar argument as the proof of Theorem 7, we have

$$\begin{aligned} & \sup_{\mu_n \in \tilde{\mathcal{P}}(n)} \int_{b, \mathbf{r}} (b - \mathbf{r}^\top \mathbf{x}(c_n, \mathcal{S}_n))^2 d\mu_n(b, \mathbf{r}) \\ & \leq \frac{1}{n} \sum_{i=1}^n (b_i - \mathbf{r}_i^\top \mathbf{x}(\mathbb{P}))^2 + c_n^2 (1 + \|\mathbf{x}(\mathbb{P})\|_1)^2 \\ & \quad + 2c_n (1 + \|\mathbf{x}(\mathbb{P})\|_1) \sqrt{\frac{1}{n} \sum_{i=1}^n (b_i - \mathbf{r}_i^\top \mathbf{x}(\mathbb{P}))^2}, \end{aligned} \tag{11}$$

and

$$\begin{aligned} & \int_{b, \mathbf{r}} (b - \mathbf{r}^\top \mathbf{x}(c_n, \mathcal{S}_n))^2 d\mathbb{P}(b, \mathbf{r}) \\ & \leq \inf_{\mu_n \in \tilde{\mathcal{P}}(n)} \left\{ \int_{b, \mathbf{r}} (b - \mathbf{r}^\top \mathbf{x}(c_n, \mathcal{S}_n))^2 d\mu_n(b, \mathbf{r}) \right. \\ & \quad \left. + \max_{b, \mathbf{r}} (b - \mathbf{r}^\top \mathbf{x}(c_n, \mathcal{S}_n))^2 \int_{b, \mathbf{r}} |f_{\mu_n}(b, \mathbf{r}) - f(b, \mathbf{r})| d(b, \mathbf{r}) \right\} \\ & \leq \sup_{\mu_n \in \tilde{\mathcal{P}}(n)} \int_{b, \mathbf{r}} (b - \mathbf{r}^\top \mathbf{x}(c_n, \mathcal{S}_n))^2 d\mu_n(b, \mathbf{r}) \\ & \quad + 2C/c_n^2 \inf_{\mu'_n \in \tilde{\mathcal{P}}(n)} \left\{ \int_{b, \mathbf{r}} |f_{\mu'_n}(b, \mathbf{r}) - f(b, \mathbf{r})| d(b, \mathbf{r}) \right\}, \end{aligned}$$

here f_μ stands for the density function of a measure μ . Notice that $\tilde{\mathcal{P}}_n$ is the set of distributions satisfying Equation (10), hence $\inf_{\mu'_n \in \tilde{\mathcal{P}}(n)} \int_{b, \mathbf{r}} |f_{\mu'_n}(b, \mathbf{r}) - f(b, \mathbf{r})| d(b, \mathbf{r})$ is upper-bounded by $\sum_{t=1}^T |\mathbb{P}(V_t) - \#(b_i, \mathbf{r}_i \in V_t)|/n$, which goes to zero as n increases for any fixed c_n (see for example Proposition A6.6 of [30]). Therefore,

$$2C/c_n^2 \inf_{\mu'_n \in \tilde{\mathcal{P}}(n)} \left\{ \int_{b, \mathbf{r}} |f_{\mu'_n}(b, \mathbf{r}) - f(b, \mathbf{r})| d(b, \mathbf{r}) \right\} \rightarrow 0,$$

if $c_n \downarrow 0$ sufficiently slow. Combining this with Inequality (11) proves the theorem. \blacksquare

VI. STABILITY

Knowing that the robust regression problem (1) and in particular Lasso encourage sparsity, it is of interest to investigate another desirable characteristic of a learning algorithm, namely, stability. We show in this section that Lasso *is not stable*. This is a special case of a more general result we prove in [31], where we show that this is a common property for all algorithms that encourage sparsity. That is, if a learning algorithm achieves certain sparsity condition, then it cannot have a non-trivial stability bound.

We recall the definition of uniform stability [29] first. We let \mathcal{Z} denote the space of points and labels (typically this will be a compact subset of \mathbb{R}^{n+1}) so that $S \in \mathcal{Z}^m$ denotes a collection of m labelled training points. We let \mathbb{L} denote a learning algorithm, and for $S \in \mathcal{Z}^m$, we let \mathbb{L}_S

denote the output of the learning algorithm (i.e., the regression function it has learned from the training data). Then given a loss function l , and a labeled point $s = (\mathbf{z}, b) \in \mathcal{Z}$, we let $l(\mathbb{L}_S, s)$ denote the loss of the algorithm that has been trained on the set S , on the data point s . Thus for squared loss, we would have $l(\mathbb{L}_S, s) = \|\mathbb{L}_S(\mathbf{z}) - b\|_2$.

Definition 1: An algorithm \mathbb{L} has uniform stability bound of β_m with respect to the loss function l if the following holds

$$\forall S \in \mathcal{Z}^m, \forall i \in \{1, \dots, m\}, \|l(\mathbb{L}_S, \cdot) - l(\mathbb{L}_{S \setminus i}, \cdot)\|_\infty \leq \beta_m.$$

Here $\mathbb{L}_{S \setminus i}$ stands for the learned solution with the i^{th} sample removed from S .

At first glance, this definition may seem too stringent for any reasonable algorithm to exhibit good stability properties. However, as shown in [29], *Tikhonov-regularized regression has stability that scales as $1/m$* . Stability that scales at least as fast as $o(\frac{1}{\sqrt{m}})$ can be used to establish strong PAC bounds (see [29]).

In this section we show that not only is the stability (in the sense defined above) of Lasso much worse than the stability of ℓ^2 -regularized regression, but in fact Lasso's stability is, in the following sense, as bad as it gets. To this end, we define the notion of the trivial bound, which is the worst possible error a training algorithm can have for arbitrary training set and testing sample labelled by zero.

Definition 2: Given a subset from which we can draw m labelled points, $\mathcal{Z} \subseteq \mathbb{R}^{n \times (m+1)}$ and a subset for one unlabelled point, $\mathcal{X} \subseteq \mathbb{R}^m$, a trivial bound for a learning algorithm \mathbb{L} w.r.t. \mathcal{Z} and \mathcal{X} is

$$\mathfrak{b}(\mathbb{L}, \mathcal{Z}, \mathcal{X}) \triangleq \max_{S \in \mathcal{Z}, \mathbf{z} \in \mathcal{X}} l(\mathbb{L}_S, (\mathbf{z}, 0)).$$

As above, $l(\cdot, \cdot)$ is a given loss function.

Notice that the trivial bound does not diminish as the number of samples increases, since by repeatedly choosing the worst sample, the algorithm will yield the same solution.

Now we show that the uniform stability bound of Lasso can be no better than its trivial bound with the number of features halved.

Theorem 9: Given $\hat{\mathcal{Z}} \subseteq \mathbb{R}^{n \times (2m+1)}$ be the domain of sample set and $\hat{\mathcal{X}} \subseteq \mathbb{R}^{2m}$ be the domain of new observation, such that

$$\begin{aligned} (\mathbf{b}, A) \in \mathcal{Z} &\implies (\mathbf{b}, A, A) \in \hat{\mathcal{X}}, \\ (\mathbf{z}^\top) \in \mathcal{X} &\implies (\mathbf{z}^\top, \mathbf{z}^\top) \in \hat{\mathcal{X}}. \end{aligned}$$

Then the uniform stability bound of Lasso is lower bounded by $\mathfrak{b}(\text{Lasso}, \mathcal{Z}, \mathcal{X})$.

Proof: Let (\mathbf{b}^*, A^*) and $(0, \mathbf{z}^{*\top})$ be the sample set and the new observation such that they jointly achieve $\mathfrak{b}(\text{Lasso}, \mathcal{Z}, \mathcal{X})$, and let \mathbf{x}^* be the optimal solution to Lasso w.r.t. (\mathbf{b}^*, A^*) . Consider the following sample set

$$\begin{pmatrix} \mathbf{b}^* & A^* & A^* \\ 0 & \mathbf{0}^\top & \mathbf{z}^{*\top} \end{pmatrix}.$$

Observe that $(\mathbf{x}^\top, \mathbf{0}^\top)^\top$ is an optimal solution of Lasso w.r.t to this sample set. Now remove the last sample from the sample set. Notice that $(\mathbf{0}^\top, \mathbf{x}^\top)^\top$ is an optimal solution for this new sample set. Using the last sample as a testing observation, the solution w.r.t the full sample set has zero cost, while the solution of the leave-one-out sample set has a cost $\mathfrak{b}(\text{Lasso}, \mathcal{Z}, \mathcal{X})$. And hence we prove the theorem. \blacksquare

VII. CONCLUSION

In this paper, we considered robust regression with a least-square-error loss. In contrast to previous work on robust regression, we considered the case where the perturbations of the observations are in the features. We show that this formulation is equivalent to a weighted ℓ^1 norm regularized regression problem if no correlation of disturbances among different features is allowed, and hence provide an interpretation of the widely used Lasso algorithm from a robustness perspective. We also formulated tractable robust regression problems for disturbance coupled among different features and hence generalize Lasso to a wider class of regularization schemes.

The sparsity and consistency of Lasso are also investigated based on its robustness interpretation. In particular we present a “no-free-lunch” theorem saying that sparsity and algorithmic stability contradict each other. This result shows, although sparsity and algorithmic stability are both regarded as desirable properties of regression algorithms, it is not possible to achieve them simultaneously, and we have to tradeoff these two properties in designing a regression algorithm.

The main thrust of this work is to treat the widely used regularized regression scheme from a robust optimization perspective, and extend the result of [13] (i.e., Tikhonov regularization is equivalent to a robust formulation for Frobenius norm bounded disturbance set) to a broader range of disturbance set and hence regularization scheme. This provides us not only with new insight of why regularization schemes work, but also offer solid motivations for selecting regularization parameter for existing regularization scheme and facilitate designing new regularizing schemes.

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APPENDIX A

PROOF OF THEOREM 2

Theorem 2. Consider a random vector $\mathbf{v} \in \mathbb{R}^n$, such that $\mathbb{E}(\mathbf{v}) = \mathbf{a}$, and $\mathbb{E}(\mathbf{v}\mathbf{v}^\top) = \Sigma$, $\Sigma \succeq 0$. Then we have

$$\Pr\{\|\mathbf{v}\|_2 \geq c_i\} \leq \begin{cases} \min_{P, \mathbf{q}, r, \lambda} & \text{Trace}(\Sigma P) + 2\mathbf{q}^\top \mathbf{a} + r \\ \text{subject to:} & \begin{pmatrix} P & \mathbf{q} \\ \mathbf{q}^\top & r \end{pmatrix} \succeq 0 \\ & \begin{pmatrix} I(m) & \mathbf{0} \\ \mathbf{0}^\top & -c_i^2 \end{pmatrix} \preceq \lambda \begin{pmatrix} P & \mathbf{q} \\ \mathbf{q}^\top & r - 1 \end{pmatrix} \\ & \lambda \geq 0. \end{cases} \quad (12)$$

Proof: Consider a function $f(\cdot)$ parameterized by P, \mathbf{q}, r defined as $f(\mathbf{v}) = \mathbf{v}^\top P \mathbf{v} + 2\mathbf{q}^\top \mathbf{v} + r$. Notice $\mathbb{E}(f(\mathbf{v})) = \text{Trace}(\Sigma P) + 2\mathbf{q}^\top \mathbf{a} + r$. Now we show that $f(\mathbf{v}) \geq 1_{\|\mathbf{v}\|_2 \geq c_i}$ for all P, \mathbf{q}, r satisfying the constraints in (12).

To show $f(\mathbf{v}) \geq 1_{\|\mathbf{v}\|_2 \geq c_i}$, we need to establish (i) $f(\mathbf{v}) \geq 0$ for all \mathbf{v} , and (ii) $f(\mathbf{v}) \geq 1$ when $\|\mathbf{v}\|_2 \geq c_i$. Notice that

$$f(\mathbf{v}) = \begin{pmatrix} \mathbf{v} \\ 1 \end{pmatrix}^\top \begin{pmatrix} P & \mathbf{q} \\ \mathbf{q}^\top & r \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ 1 \end{pmatrix},$$

hence (i) holds because

$$\begin{pmatrix} P & \mathbf{q} \\ \mathbf{q}^\top & r \end{pmatrix} \succeq 0.$$

To establish condition (ii), it suffices to show $\mathbf{v}^\top \mathbf{v} \geq c_i^2$ implies $\mathbf{v}^\top P \mathbf{v} + 2\mathbf{q}^\top \mathbf{v} + r \geq 1$, which is equivalent to show $\{\mathbf{v} | \mathbf{v}^\top P \mathbf{v} + 2\mathbf{q}^\top \mathbf{v} + r - 1 \leq 0\} \subseteq \{\mathbf{v} | \mathbf{v}^\top \mathbf{v} \leq c_i^2\}$. Noticing this is an ellipsoid-containment condition, by S-Procedure, we see that is equivalent to the condition that

there exists a $\lambda \geq 0$ such that

$$\begin{pmatrix} I^{(m)} & \mathbf{0} \\ \mathbf{0}^\top & -c_i^2 \end{pmatrix} \preceq \lambda \begin{pmatrix} P & \mathbf{q} \\ \mathbf{q}^\top & r-1 \end{pmatrix}.$$

Hence we have $f(\mathbf{v}) \geq 1_{\|\mathbf{v}\|_2 \geq c_i}$, taking expectation over both side that notice that the expectation of a indicator function is the probability, we establish the theorem. ■

APPENDIX B PROOF OF THEOREM 4

Theorem 4. Assume that the set

$$\mathcal{Z} \triangleq \{\mathbf{z} \in \mathbb{R}^m | f_j(\mathbf{z}) \leq 0, j = 1, \dots, k; \mathbf{z} \geq \mathbf{0}\}$$

has non-empty relative interior. Then the robust regression problem

$$\min_{\mathbf{x} \in \mathbb{R}^m} \left\{ \max_{\Delta A \in \mathcal{U}'} \|\mathbf{b} - (A + \Delta A)\mathbf{x}\|_a \right\}$$

is equivalent to the following regularized regression problem

$$\min_{\lambda \in \mathbb{R}_+^k, \kappa \in \mathbb{R}_+^m, \mathbf{x} \in \mathbb{R}^m} \left\{ \|\mathbf{b} - A\mathbf{x}\|_a + v(\lambda, \kappa, \mathbf{x}) \right\};$$

where: $v(\lambda, \kappa, \mathbf{x}) \triangleq \max_{\mathbf{c} \in \mathbb{R}^m} \left[(\kappa + |\mathbf{x}|)^\top \mathbf{c} - \sum_{j=1}^k \lambda_j f_j(\mathbf{c}) \right]$

Proof: Fix a solution \mathbf{x}^* . Notice that,

$$\mathcal{U}' = \{(\delta_1, \dots, \delta_m) | \mathbf{c} \in \mathcal{Z}; \|\delta_i\|_a \leq c_i, i = 1, \dots, m\}.$$

Hence we have:

$$\begin{aligned} & \max_{\Delta A \in \mathcal{U}'} \|\mathbf{b} - (A + \Delta A)\mathbf{x}^*\|_a \\ &= \max_{\mathbf{c} \in \mathcal{Z}} \left\{ \max_{\|\delta_i\|_a \leq c_i, i=1, \dots, m} \|\mathbf{b} - (A + (\delta_1, \dots, \delta_m))\mathbf{x}^*\|_a \right\} \\ &= \max_{\mathbf{c} \in \mathcal{Z}} \left\{ \|\mathbf{b} - A\mathbf{x}^*\|_a + \sum_{i=1}^m c_i |x_i^*| \right\} \\ &= \|\mathbf{b} - A\mathbf{x}^*\|_a + \max_{\mathbf{c} \in \mathcal{Z}} \left\{ |\mathbf{x}^*|^\top \mathbf{c} \right\}. \end{aligned} \tag{13}$$

The second equation follows from Theorem 3.

Now we need to evaluate $\max_{\mathbf{c} \in \mathcal{Z}} \{|\mathbf{x}^*|^\top \mathbf{c}\}$, which equals to $-\min_{\mathbf{c} \in \mathcal{Z}} \{-|\mathbf{x}^*|^\top \mathbf{c}\}$. Hence we are minimizing a linear function over a set of convex constraints. Furthermore, by assumption the Slater's condition holds. Hence the duality gap of $\min_{\mathbf{c} \in \mathcal{Z}} \{-|\mathbf{x}^*|^\top \mathbf{c}\}$ is zero. A standard duality analysis shows that

$$\max_{\mathbf{c} \in \mathcal{Z}} \left\{ |\mathbf{x}^*|^\top \mathbf{c} \right\} = \min_{\lambda \in \mathbb{R}_+^k, \kappa \in \mathbb{R}_+^m} v(\lambda, \kappa, \mathbf{x}^*). \tag{14}$$

We establish the theorem by substituting Equation (14) back into Equation (13) and taking minimum over \mathbf{x} on both sides. \blacksquare

APPENDIX C PROOF OF PROPOSITION 1

Proposition 1. *Given a function $g : \mathbb{R}^{m+1} \rightarrow \mathbb{R}$ and Borel sets $\mathcal{Z}_1, \dots, \mathcal{Z}_n \subseteq \mathbb{R}^{m+1}$, let*

$$\mathcal{P}_n \triangleq \{\mu \in \mathcal{P} \mid \forall S \subseteq \{1, \dots, n\} : \mu(\bigcup_{i \in S} \mathcal{Z}_i) \geq |S|/n\}.$$

The following holds

$$\frac{1}{n} \sum_{i=1}^n \sup_{(\mathbf{r}_i, b_i) \in \mathcal{Z}_i} h(\mathbf{r}_i, b_i) = \sup_{\mu \in \mathcal{P}_n} \int_{\mathbb{R}^{m+1}} h(\mathbf{r}, b) d\mu(\mathbf{r}, b).$$

Proof: To prove Proposition 1, we first establish the following lemma.

Lemma 2: *Given a function $f : \mathbb{R}^{m+1} \rightarrow \mathbb{R}$, and a Borel set $\mathcal{Z} \subseteq \mathbb{R}^{m+1}$, the following holds:*

$$\sup_{\mathbf{x}' \in \mathcal{Z}} f(\mathbf{x}') = \sup_{\mu \in \mathcal{P} \mid \mu(\mathcal{Z})=1} \int_{\mathbb{R}^{m+1}} f(\mathbf{x}) d\mu(\mathbf{x}).$$

Proof: Let $\hat{\mathbf{x}}$ be a ϵ -optimal solution to the left hand side, consider the probability measure μ' that put mass 1 on $\hat{\mathbf{x}}$, which satisfy $\mu'(\mathcal{Z}) = 1$. Hence, we have

$$\sup_{\mathbf{x}' \in \mathcal{Z}} f(\mathbf{x}') - \epsilon \leq \sup_{\mu \in \mathcal{P} \mid \mu(\mathcal{Z})=1} \int_{\mathbb{R}^{m+1}} f(\mathbf{x}) d\mu(\mathbf{x}),$$

since ϵ can be arbitrarily small, this leads to

$$\sup_{\mathbf{x}' \in \mathcal{Z}} f(\mathbf{x}') \leq \sup_{\mu \in \mathcal{P} \mid \mu(\mathcal{Z})=1} \int_{\mathbb{R}^{m+1}} f(\mathbf{x}) d\mu(\mathbf{x}). \quad (15)$$

Next construct function $\hat{f} : \mathbb{R}^{m+1} \rightarrow \mathbb{R}$ as

$$\hat{f}(\mathbf{x}) \triangleq \begin{cases} f(\hat{\mathbf{x}}) & \mathbf{x} \in \mathcal{Z}; \\ f(\mathbf{x}) & \text{otherwise.} \end{cases}$$

By definition of $\hat{\mathbf{x}}$ we have $f(\mathbf{x}) \leq \hat{f}(\mathbf{x}) + \epsilon$ for all $\mathbf{x} \in \mathbb{R}^{m+1}$. Hence, for any probability measure μ such that $\mu(\mathcal{Z}) = 1$, the following holds

$$\int_{\mathbb{R}^{m+1}} f(\mathbf{x}) d\mu(x) \leq \int_{\mathbb{R}^{m+1}} \hat{f}(\mathbf{x}) d\mu(x) + \epsilon = f(\hat{\mathbf{x}}) + \epsilon \leq \sup_{\mathbf{x}' \in \mathcal{Z}} f(\mathbf{x}') + \epsilon.$$

This leads to

$$\sup_{\mu \in \mathcal{P} \mid \mu(\mathcal{Z})=1} \int_{\mathbb{R}^{m+1}} f(\mathbf{x}) d\mu(x) \leq \sup_{\mathbf{x}' \in \mathcal{Z}} f(\mathbf{x}') + \epsilon.$$

Notice ϵ can be arbitrarily small, we have

$$\sup_{\mu \in \mathcal{P} \mid \mu(\mathcal{Z})=1} \int_{\mathbb{R}^{m+1}} f(\mathbf{x}) d\mu(x) \leq \sup_{\mathbf{x}' \in \mathcal{Z}} f(\mathbf{x}') \quad (16)$$

Combining (15) and (16), we prove the lemma. \blacksquare

Now we proceed to prove the proposition. Let $\hat{\mathbf{x}}_i$ be an ϵ -optimal solution to $\sup_{\mathbf{x}_i \in \mathcal{Z}_i} f(\mathbf{x}_i)$. Observe that the empirical distribution for $(\hat{\mathbf{x}}_1, \dots, \hat{\mathbf{x}}_n)$ belongs to \mathcal{P}_n , since ϵ can be arbitrarily close to zero, we have

$$\frac{1}{n} \sum_{i=1}^n \sup_{\mathbf{x}_i \in \mathcal{Z}_i} f(\mathbf{x}_i) \leq \sup_{\mu \in \mathcal{P}_n} \int_{\mathbb{R}^{m+1}} f(\mathbf{x}) d\mu(\mathbf{x}). \quad (17)$$

Without loss of generality, assume

$$f(\hat{\mathbf{x}}_1) \leq f(\hat{\mathbf{x}}_2) \leq \dots \leq f(\hat{\mathbf{x}}_n). \quad (18)$$

Now construct the following function

$$\hat{f}(\mathbf{x}) \triangleq \begin{cases} \min_{i|\mathbf{x} \in \mathcal{Z}_i} f(\hat{\mathbf{x}}_i) & \mathbf{x} \in \bigcup_{j=1}^n \mathcal{Z}_j; \\ f(\mathbf{x}) & \text{otherwise.} \end{cases} \quad (19)$$

Observe that $f(\mathbf{x}) \leq \hat{f}(\mathbf{x}) + \epsilon$ for all \mathbf{x} .

Furthermore, given $\mu \in \mathcal{P}_n$, we have

$$\begin{aligned} & \int_{\mathbb{R}^{m+1}} f(\mathbf{x}) d\mu(\mathbf{x}) - \epsilon \\ &= \int_{\mathbb{R}^{m+1}} \hat{f}(\mathbf{x}) d\mu(\mathbf{x}) \\ &= \sum_{k=1}^n f(\hat{\mathbf{x}}_k) \left[\mu\left(\bigcup_{i=1}^k \mathcal{Z}_i\right) - \mu\left(\bigcup_{i=1}^{k-1} \mathcal{Z}_i\right) \right] \end{aligned}$$

Denote $\alpha_k \triangleq \left[\mu\left(\bigcup_{i=1}^k \mathcal{Z}_i\right) - \mu\left(\bigcup_{i=1}^{k-1} \mathcal{Z}_i\right) \right]$, we have

$$\sum_{k=1}^n \alpha_k = 1, \quad \sum_{k=1}^t \alpha_k \geq t/n.$$

Hence by Equation (18) we have

$$\sum_{k=1}^n \alpha_k f(\hat{\mathbf{x}}_k) \leq \frac{1}{n} \sum_{k=1}^n f(\hat{\mathbf{x}}_k).$$

Thus we have for any $\mu \in \mathcal{P}_n$,

$$\int_{\mathbb{R}^{m+1}} f(\mathbf{x}) d\mu(\mathbf{x}) - \epsilon \leq \frac{1}{n} \sum_{k=1}^n f(\hat{\mathbf{x}}_k).$$

Therefore,

$$\sup_{\mu \in \mathcal{P}_n} \int_{\mathbb{R}^{m+1}} f(\mathbf{x}) d\mu(\mathbf{x}) - \epsilon \leq \sup_{\mathbf{x}_i \in \mathcal{Z}_i} \frac{1}{n} \sum_{k=1}^n f(\mathbf{x}_k).$$

Notice ϵ can be arbitrarily close to 0, we proved the proposition by combining with (17). ■

APPENDIX D
PROOF OF COROLLARY 3

Corollary 3. Given $\mathbf{b} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times m}$, the following equation holds for any $\mathbf{x} \in \mathbb{R}^m$,

$$\|\mathbf{b} - A\mathbf{x}\|_2 + \sqrt{n}c_n\|\mathbf{x}\|_1 + \sqrt{n}c_n = \sup_{\mu \in \hat{\mathcal{P}}(n)} \sqrt{n \int_{\mathbb{R}^{m+1}} (b' - \mathbf{r}'^\top \mathbf{x})^2 d\mu(\mathbf{r}', b')}. \quad (20)$$

Here,

$$\begin{aligned} \hat{\mathcal{P}}(n) &\triangleq \bigcup_{\|\boldsymbol{\sigma}\|_2 \leq \sqrt{n}c_n; \forall i: \|\boldsymbol{\delta}_i\|_2 \leq \sqrt{n}c_n} \mathcal{P}_n(A, \Delta, \mathbf{b}, \boldsymbol{\sigma}); \\ \mathcal{P}_n(A, \Delta, \mathbf{b}, \boldsymbol{\sigma}) &\triangleq \{\mu \in \mathcal{P} | \mathcal{Z}_i = [b_i - \sigma_i, b_i + \sigma_i] \times \prod_{j=1}^m [a_{ij} - \delta_{ij}, a_{ij} + \delta_{ij}]; \\ &\quad \forall S \subseteq \{1, \dots, n\} : \mu(\bigcup_{i \in S} \mathcal{Z}_i) \geq |S|/n\}. \end{aligned}$$

Proof: The right-hand-side of Equation (20) equals

$$\sup_{\|\boldsymbol{\sigma}\|_2 \leq \sqrt{n}c_n; \forall i: \|\boldsymbol{\delta}_i\|_2 \leq \sqrt{n}c_n} \left\{ \sup_{\mu \in \mathcal{P}_n(A, \Delta, \mathbf{b}, \boldsymbol{\sigma})} \sqrt{n \int_{\mathbb{R}^{m+1}} (b' - \mathbf{r}'^\top \mathbf{x})^2 d\mu(\mathbf{r}', b')} \right\}.$$

Notice by the equivalence to robust formulation, the left-hand-side equals to

$$\begin{aligned} &\max_{\|\boldsymbol{\sigma}\|_2 \leq \sqrt{n}c_n; \forall i: \|\boldsymbol{\delta}_i\|_2 \leq \sqrt{n}c_n} \left\| \mathbf{b} + \boldsymbol{\sigma} - (A + [\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_m])\mathbf{x} \right\|_2 \\ &= \sup_{\|\boldsymbol{\sigma}\|_2 \leq \sqrt{n}c_n; \forall i: \|\boldsymbol{\delta}_i\|_2 \leq \sqrt{n}c_n} \left\{ \sup_{(\hat{b}_i, \hat{\mathbf{r}}_i) \in [b_i - \sigma_i, b_i + \sigma_i] \times \prod_{j=1}^m [a_{ij} - \delta_{ij}, a_{ij} + \delta_{ij}]} \sqrt{\sum_{i=1}^n (\hat{b}_i - \hat{\mathbf{r}}_i^\top \mathbf{x})^2} \right\} \\ &= \sup_{\|\boldsymbol{\sigma}\|_2 \leq \sqrt{n}c_n; \forall i: \|\boldsymbol{\delta}_i\|_2 \leq \sqrt{n}c_n} \sqrt{\sum_{i=1}^n \sup_{(\hat{b}_i, \hat{\mathbf{r}}_i) \in [b_i - \sigma_i, b_i + \sigma_i] \times \prod_{j=1}^m [a_{ij} - \delta_{ij}, a_{ij} + \delta_{ij}]} (\hat{b}_i - \hat{\mathbf{r}}_i^\top \mathbf{x})^2}, \end{aligned}$$

furthermore, applying Proposition 1 yields

$$\begin{aligned} &\sqrt{\sum_{i=1}^n \sup_{(\hat{b}_i, \hat{\mathbf{r}}_i) \in [b_i - \sigma_i, b_i + \sigma_i] \times \prod_{j=1}^m [a_{ij} - \delta_{ij}, a_{ij} + \delta_{ij}]} (\hat{b}_i - \hat{\mathbf{r}}_i^\top \mathbf{x})^2} \\ &= \sqrt{\sup_{\mu \in \mathcal{P}_n(A, \Delta, \mathbf{b}, \boldsymbol{\sigma})} n \int_{\mathbb{R}^{m+1}} (b' - \mathbf{r}'^\top \mathbf{x})^2 d\mu(\mathbf{r}', b')} \\ &= \sup_{\mu \in \mathcal{P}_n(A, \Delta, \mathbf{b}, \boldsymbol{\sigma})} \sqrt{n \int_{\mathbb{R}^{m+1}} (b' - \mathbf{r}'^\top \mathbf{x})^2 d\mu(\mathbf{r}', b')}, \end{aligned}$$

which proves the corollary. ■